The Super-Liouville Equation on the Half-Line

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Abstract

A recursive formula for an infinity of integrals of motion for the super-Liouville theory is derived. The integrable boundary interactions for this theory and the super-Toda theory based on the affine superalgebra $B^{(1)}(0,1)$ are computed. In the first case the boundary interactions are unambiguously determined by supersymmetry, whilst in the latter case there are free parameters.

1 Introduction

The study of boundary quantum integrable models has a wide range of applications, notably open string theory and dissipative quantum mechanics [10] [15]. Considerable progress has been made with the pioneering work of Ghoshal and Zamolodchikov [11], where they computed the boundary S-matrix for the sine-Gordon theory.

In the past few years, there has been renewed interest in the problem of incorporating fermions in Toda field theories, [1] - [3]. With the exceptions of the Liouville and sinh-Gordon theories supersymmetrizing bosonic Toda models is not a simple matter. If one focusses attention on the integrability of the models, rather than supersymmetry, it is possible to construct a new class of Toda models with fermions, where the underlying algebra is a Lie superalgebra, [4] [5]. Following the work of Zamolodchikov [6] [7] some attempts were made at determining exact S-matrices for this class of theories [8] [9].

Recently, Inami et al. [12] considered the supersymmetric extension of the sine-Gordon theory on the half-line and found that the requirements of integrability and supersymmetry fully determine the boundary potential up to an overall sign.

In this paper, we apply similar considerations to the super-Liouville theory and the Toda theory based on the superalgebra $B^{(1)}(0,1)$.

Besides its applications in statistical mechanics, the super-Liouville equation (SLE) arises in Polyakov's approach to the superstring, [13]. As a conformal field theory, the SLE can be described by its integrable structure along with its more conventional characterization in terms of the Virasoro algebra and its representations (e.g. [16] and references therein). In fact, an infinite set of involutive integrals of motion (IM) can be shown to exist. The IM are just composite fields of the stress tensor, the supercurrents and their derivatives. The theory is then characterized by the massless states diagonalizing the IM and their factorizable S-matrix. We will show that these IM can be derived using Lax pair techniques. The boundary conditions will be determined by requiring the preservation of the superconformal invariance, [11] [14] [15]. As in the case of the super-sine-Gordon theory [12], this strongly restricts the boundary equations of motion, so that there will be no free parameters whatsoever. Furthermore, these conditions equally allow for the conservation of half of the IM, making it still possible to approach the theory on the half-line from the point of view of the boundary massless S-matrix.

The $B^{(1)}(0,1)$ theory on the other hand is massive and the determination of the boundary potential will rely on the premiss that its form can be conjectured with generality by preserving certain combinations of the lower spin charges, [11] [12]. However, this theory is not supersymmetric and there will thus be free parameters in the boundary potential.

This paper is organised as follows. In section 2 we define the theories in the bulk. We discuss superconformal invariance and derive the IM for the SLE. We also compute the spin 1 and 3 densities of the $B^{(1)}(0,1)$ theory. In section 3 we establish the boundary conditions for both theories. Finally, we summarize our results in section 4.

2 The $B(0,1)$ and $B^{(1)}(0,1)$ theories

In this section, we define the theories in the bulk. For the SLE, we discuss supersymmetry and determine a recursive formula for the IM using a method developed in refs. [17], [18]. This formula is a supersymmetrized version of the IM of ref. [16] to which it reduces in the bosonic limit. We also derive the spin 1 and 3 conserved densities for the $B^{(1)}(0,1)$

theory by considering the most general Ansatz. Bearing in mind that our discussion is strictly classical, the above ingredients should suffice to construct the integrable boundary interactions with considerable generality.

Firstly, we establish our notation. Consider two-dimensional superspace, with units such that $\hbar = c = 1$ and $[mass] = 1$, and the superspace coordinate

$$Z = (x_\mu, \theta_A) = (x_0, x_1, \theta_1, \theta_2),$$

where x_μ is the coordinate on two-dimensional Minkowski space and θ_A are Grassmann variables. We introduce the scalar superfield Φ with components:

$$\Phi = \varphi + i\theta_1\psi_2 - i\theta_2\psi_1 + \theta_1\theta_2 F.$$

The superderivatives

$$D_1 = -\partial_{\theta_2} + i\theta_2\partial_+, \qquad D_2 = \partial_{\theta_1} + i\theta_1\partial_-$$

have the properties

$$D_1^2 = -i\partial_+, \qquad D_2^2 = i\partial_-,$$

where the light-cone variables are defined as

$$x_\pm \equiv \frac{x_1 \pm x_0}{2}.$$

2.1   The super-Liouville theory

Let us define the following linear system:

$$\begin{cases} D_1\chi = A_1(\lambda)\chi \\ D_2\chi = A_2(\lambda)\chi \end{cases} \tag{1}$$

χ is a column vector, whose components are the bosonic superfields V_1, V_2 and the fermionic superfield V_3; λ is an arbitrary parameter with dimension of mass, and A_1, A_2 are the graded matrices:

$$A_1(\lambda) = -\sqrt{\frac{2}{\lambda}}\begin{pmatrix} 0 & 0 & e^{2\Phi} \\ 0 & 0 & ie^{2\Phi} \\ e^{2\Phi} & ie^{2\Phi} & 0 \end{pmatrix}; \qquad A_2(\lambda) = \begin{pmatrix} \lambda\theta_1 & -2iD_2\Phi & 0 \\ 2iD_2\Phi & \lambda\theta_1 & -\sqrt{2\lambda} \\ 0 & \sqrt{2\lambda} & \lambda\theta_1 \end{pmatrix}.$$

The integrability condition for the system (1) is just the $N = 1$ SLE,

$$D_1D_2\Phi = ie^{2\Phi}, \tag{2}$$

which is the simplest example of a Toda theory based on a contragradient Lie superalgebra. This superalgebra is labelled $B(0,1)$ in the calssification of Kac [4] and it possesses three bosonic generators and two fermionic ones. A realization of $B(0,1)$ is provided by $Osp(1|2;\mathcal{C})$. The theory based on this finite superalgebra is conformally invariant. Furthermore, the SLE also happens to be supersymmetric and therefore superconformal, [13].

Notice that eq.(2) is independent of the spectral parameter λ. This will give rise to an infinity of conservation laws.

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Writing out eq.(2) in components, we have:

$$\begin{cases} F = -ie^{2\varphi}; \qquad \partial_-\psi_1 = -2e^{2\varphi}\psi_2; \qquad \partial_+\psi_2 = -2e^{2\varphi}\psi_1 \\ \partial_+\partial_-\varphi = 2e^{2\varphi}(e^{2\varphi} + 2i\psi_1\psi_2) \end{cases} \tag{3}$$

The above equations of motion can be derived from the superspace action

$$S = \frac{1}{2}\int d^2z\,d^2\theta(D_1\Phi D_2\Phi + ie^{2\Phi}).$$

We now define two new scalar superfields U, Z and a fermionic superfield Y as:

$$U = lnV_1 + i\lambda x_-; \qquad Z = \frac{V_2}{V_1}; \qquad Y = \frac{V_3}{V_1}.$$

We then have:

$$D_1U = -\sqrt{\tfrac{2}{\lambda}}e^{2\Phi}Y; \qquad D_2U = -2iD_2\Phi \cdot Z$$

$$\sqrt{2\lambda}Y = 2iD_2\Phi - D_2Z + 2iD_2\Phi \cdot Z \cdot Z \tag{4}$$

$$\sqrt{2\lambda}Z = D_2Y - 2iD_2\Phi \cdot Z \cdot Y$$

Taking into account that $Y^2 = (D_2\Phi)^2 = 0$, we get the following differential equation for Y:

$$\sqrt{2\lambda}Y = 2iD_2\Phi - \frac{i}{\sqrt{2\lambda}}\partial_-Y - \frac{1}{\lambda}\partial_-\Phi \cdot D_2Y \cdot Y. \tag{5}$$

We assume an expansion of Y in a power series of λ^{-1}:

$$Y = \frac{1}{i\sqrt{2\lambda}}\sum_{n=0}^{\infty}\frac{Y^{(n+1/2)}}{(2i\lambda)^n}. \tag{6}$$

Substituting this expansion in eq.(5) and equating powers of λ^{-1}, we obtain the following recursive formula:

$$\begin{aligned} &Y^{(1/2)} = -2D_2\Phi; \\ &Y^{(n+1/2)} = \partial_-Y^{(n-1/2)} - 2i\partial_-\Phi \cdot \sum_{l=1}^{n-1}D_2Y^{(l-1/2)} \cdot Y^{(n-l-1/2)}; \\ &n = 1, 2, 3, \cdots \end{aligned} \tag{7}$$

The integrability condition,

$$D_1D_2U = -D_2D_1U,$$

can be interpreted as an infinite number of supersymmetric covariant conservation laws:

$$D_1J_2^{(n+1/2)} = D_2J_1^{(n+1/2)}; \qquad n = 1, 2, 3, \cdots, \tag{8}$$

where

$$\begin{cases} J_1^{(n+1/2)} = -e^{2\Phi} \cdot Y^{(n-1/2)} \\ J_2^{(n+1/2)} = iD_2\Phi \cdot D_2Y^{(n-1/2)} \end{cases} \tag{9}$$

One can check that the bosonic conserved quantities will be given by the $\theta_1\theta_2$ component of eq.(8). We will henceforth work in Euclidean space,

$$\begin{cases} x = x_1 \\ y = ix_0 \end{cases} \qquad \begin{cases} z = x + iy \\ \bar{z} = x - iy \end{cases}$$

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and redefine the fields,

$$\varphi = \phi/2, \qquad \psi_1 = \alpha\bar{\psi}, \qquad \psi_2 = \alpha\psi,$$

for future convenience. The parameter α is such that $\alpha^2 = i/2$. The eqsuations of motion (3) then become

$$\begin{cases} F = -ie^\phi; & \partial_z\bar{\psi} = -e^\phi\psi; \qquad \partial_{\bar{z}}\psi = -e^\phi\bar{\psi} \\ \partial_z\partial_{\bar{z}}\phi = e^{2\phi} - e^\phi\bar{\psi}\psi \end{cases} \tag{10}$$

The bosonic conservation laws are expressed in the form:

$$\partial_{\bar{z}}T_{s+1} = \partial_z\Theta_{s-1}; \qquad s = 1, 3, 5, \cdots,$$

where s is the spin of the conserved charge. Here are some elements of this sequence, which will be useful later:

$$T_2 = (\partial_z\phi)^2 - \partial_z\psi\psi$$

$$T_4 = (\partial_z^2\phi)^2 + (\partial_z\phi)^4 + 3(\partial_z\phi)^2\psi\partial_z\psi + \partial_z\psi\partial_z^2\psi$$

$$\begin{aligned} T_6 = & (\partial_z\phi)^6 - \tfrac{1}{2}\partial_z^2\phi\partial_z^4\phi - \tfrac{11}{2}(\partial_z\phi)^2(\partial_z^2\phi)^2 - \tfrac{7}{2}(\partial_z\phi)^3\partial_z^3\phi + (\partial_z\phi)^2\partial_z^2\psi\partial_z\psi + \\ & + 8\partial_z\phi\partial_z^3\phi\partial_z\psi\psi - 10\partial_z\phi\partial_z^2\phi\psi\partial_z^2\psi - \tfrac{7}{2}(\partial_z\phi)^2\psi\partial_z^3\psi + 5(\partial_z\phi)^4\psi\partial_z\psi + \\ & + \tfrac{1}{2}\partial_z^4\psi\partial_z\psi - \tfrac{11}{2}(\partial_z^2\phi)^2\psi\partial_z\psi \end{aligned}$$

$$\Theta_0 = e^{2\phi} - e^\phi\bar{\psi}\psi$$

$$\Theta_2 = 2(\partial_z\phi)^2e^{2\phi} + e^{2\phi}\psi\partial_z\psi - \partial_z\phi e^\phi\bar{\psi}\partial_z\psi - (\partial_z\phi)^2e^\phi\bar{\psi}\psi$$

$$\begin{aligned} \Theta_4 = & -\partial_z\phi\partial_z^3\phi e^{2\phi} - \tfrac{17}{2}(\partial_z\phi)^2\partial_z^2\phi e^{2\phi} - 4(\partial_z\phi)^4e^{2\phi} + \tfrac{1}{2}\partial_z\phi\partial_z^3\phi e^\phi\bar{\psi}\psi + \\ & + \tfrac{11}{2}\partial_z\phi\partial_z^2\phi e^\phi\bar{\psi}\partial_z\psi - \tfrac{3}{2}(\partial_z\phi)^2\partial_z^2\phi e^\phi\bar{\psi}\psi - \tfrac{21}{2}(\partial_z\phi)^2e^{2\phi}\psi\partial_z\psi + \\ & + \tfrac{3}{2}(\partial_z\phi)^3e^\phi\bar{\psi}\partial_z\psi + 3(\partial_z\phi)^2e^\phi\bar{\psi}\partial_z^2\psi - \tfrac{11}{2}\partial_z\phi e^{2\phi}\psi\partial_z^2\psi - \tfrac{1}{2}e^{2\phi}\psi\partial_z^3\psi + \\ & + \tfrac{1}{2}\partial_z\phi e^\phi\bar{\psi}\partial_z^3\psi - \tfrac{11}{2}\partial_z^2\phi e^{2\phi}\psi\partial_z\psi - (\partial_z\phi)^4e^\phi\bar{\psi}\psi \end{aligned}$$

These coincide with the results of ref.[19], which were obtained by using Bäcklund transformations. Note that the system (10) is invariant under $z \leftrightarrow \bar{z}$ and $\psi \to i\bar{\psi}$, $\bar{\psi} \to i\psi$, so that there will be a corresponding set of conserved quantities,

$$\partial_z\bar{T}_{s+1} = \partial_{\bar{z}}\bar{\Theta}_{s-1}, \qquad s = 1, 3, 5, \cdots$$

Even spin densities do not appear in the above sequence, because the corresponding charges vanish. To see this, we note that eq.(8) remains unchanged under the 'gauge' transformation,

$$\begin{cases} J_1^{(n+1/2)} \to J_1^{(n+1/2)} + D_1V^{(n+1/2)} \\ J_2^{(n+1/2)} \to J_2^{(n+1/2)} - D_2V^{(n+1/2)} \end{cases} \tag{11}$$

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where $V^{(n+1/2)}$ is an arbitrary scalar superfield. It is then straightforward to check that for the choices

$$V^{(3/2)} = (\partial_-\varphi)^2 + 2i\theta_1\partial_-\varphi\partial_-\psi_2 + 4i\theta_2\partial_-\varphi e^{2\varphi}\psi_2 - 4\theta_1\theta_2 e^{2\varphi}(\psi_2\partial_-\psi_2 + i(\partial_-\varphi)^2)$$

and

$$V^{(7/2)} = 2(\partial_-\varphi)^4 + (\partial_-^2\varphi)^2 - 2i\partial_-\psi_2\partial_-^2\psi_2 - 16i(\partial_-\varphi)^2\psi_2\partial_-\psi_2 +$$
$$+ i\theta_1(16(\partial_-\varphi)^2\partial_-^2\varphi\psi_2 + 2\partial_-^3\varphi\partial_-\psi_2 - 8(\partial_-\varphi)^3\partial_-\psi_2) +$$
$$+ i\theta_2 e^{2\varphi}(8\partial_-\varphi\partial_-^2\varphi\psi_2 + 8(\partial_-\varphi)^2\partial_-\psi_2 + 4\partial_-\varphi\partial_-^2\psi_2 - 16(\partial_-\varphi)^3\psi_2) +$$
$$+ \theta_1\theta_2 e^{2\varphi}(16i(\partial_-\varphi)^4 - 36i(\partial_-\varphi)^2\partial_-^2\varphi - 8\partial_-^2\varphi\psi_2\partial_-\psi_2 + 64(\partial_-\varphi)^2\psi_2\partial_-\psi_2 + 4\partial_-\psi_2\partial_-^2\partial_-^2\psi_2),$$

we get up to a transformation (11),

$$J_A^{(5/2)} = \partial_- J_A^{(3/2)}$$
$$J_A^{(9/2)} = \partial_- J_A^{(7/2)}, \qquad (A = 1, 2)$$

and the charges are thus trivial. This is consistent with the 'spin assignment' property discussed in ref. [16]. T_2, $\bar{T}_2$ and $\Theta_0 = \bar{\Theta}_0$ are the components of the stress tensor. From the equations of motion it is easy to show that Θ_0 is just $\Theta_0 = \partial_z\partial_{\bar{z}}\phi$. We then have $\partial_{\bar{z}}T = 0$, where,

$$T = T_2 - \partial_z^2\phi = (\partial_z\phi)^2 - \partial_z\psi\psi - \partial_z^2\phi. \tag{12}$$

Similarly,

$$\bar{T} = (\partial_{\bar{z}}\phi)^2 + \partial_{\bar{z}}\bar{\psi}\bar{\psi} - \partial_{\bar{z}}^2\phi. \tag{13}$$

This is just the 'conformally improved' stress tensor, [13]. The total derivative terms restore the tracelessness of the stress tensor, which is a necessary requirement for the theory to be conformally invariant.

Besides being integrable, the theory on the full line is also invariant under the supersymmetry transformations,

$$\begin{cases} \delta_S\phi = \eta\psi + \bar{\eta}\bar{\psi} \\ \delta_S\psi = -(\eta\partial_z\phi + \bar{\eta}e^\phi) \\ \delta_S\bar{\psi} = \bar{\eta}\partial_{\bar{z}}\phi + \eta e^\phi \end{cases} \tag{14}$$

where η and $\bar{\eta}$ are infinitesimal constant fermionic parameters. To see this, let us first rewrite the action in Euclidean space after eliminating the non-dynamical auxiliary field F:

$$\mathcal{L}_0 = 2(\partial_z\phi\partial_{\bar{z}}\phi + \psi\partial_{\bar{z}}\psi - \bar{\psi}\partial_z\bar{\psi} + e^{2\phi} - 2e^\phi\bar{\psi}\psi). \tag{15}$$

As expected the variation of $\mathcal{L}_0$ under these transformations then amounts to a total derivative:

$$\delta_S\mathcal{L}_0 = 2\partial_z(\delta_S\phi\partial_{\bar{z}}\phi - \bar{\psi}\delta_S\bar{\psi} - 2\partial_{\bar{z}}\phi\bar{\eta}\bar{\psi} - 2\eta\bar{\psi}e^\phi) +$$
$$+ 2\partial_{\bar{z}}(\delta_S\phi\partial_z\phi + \psi\delta_S\psi - 2\partial_z\phi\eta\psi - 2\bar{\eta}\psi e^\phi). \tag{16}$$

The infinitesimal transformations (14) are generated by the currents

$$\begin{cases} J = \psi\partial_z\phi - \partial_z\psi \\ \bar{J} = \bar{\psi}\partial_{\bar{z}}\phi - \partial_{\bar{z}}\bar{\psi} \end{cases} \tag{17}$$

These currents were also conformally improved by adding total derivative terms by hand. The fact that a conformal improvement is equally necessary for these currents is related

to the fact that they are supersymmetric partners of the stress tensor and are therefore expected to have a similar behaviour, [13]. We note that this ensures that the theory be 'chiral'. This also means that we should be able to re-express the conservation laws derived above in the form, $\partial_{\bar{z}}U_{s+1} = 0$. We have already shown that this is true for $s = 1$, where U_2 is the conformally improved stress tensor (12). Similarly, we have

$$\Theta_2 = \partial_{\bar{z}}(2/3(\partial_z\phi)^3 + \partial_z\phi\psi\partial_z\psi),$$

which means that $\partial_{\bar{z}}U_4 = 0$, where

$$U_4 = (\partial_z^2\phi)^2 + (\partial_z\phi)^4 + 3(\partial_z\phi)^2\psi\partial_z\psi + \partial_z\psi\partial_z^2\psi - 2(\partial_z\phi)^2\partial_z^2\phi - \partial_z^2\phi\psi\partial_z\psi - \partial_z\phi\psi\partial_z^2\psi.$$

We can take the bosonic limit by setting the fermionic fields to zero and we get $U_4^{(b)} = (T^{(b)})^2$, where $T^{(b)}$ is the bosonic stress tensor $T^{(b)} = (\partial_z\phi)^2 - \partial_z^2\phi$. This result motivates us to write $U_4 = T^2 + \delta U_4$, with:

$$\delta U_4 = (\partial_z\phi)^2\psi\partial_z\psi + \partial_z^2\phi\psi\partial_z\psi + \partial_z\psi\partial_z^2\psi - \partial_z\phi\psi\partial_z^2\psi.$$

The two terms are separately conserved, i.e. $\partial_{\bar{z}}T^2 = \partial_{\bar{z}}\delta U_4 = 0$. δU_4 should therefore be expressible in terms of the supercurrent (17) and its derivatives. Using dimensional conditerations, we obtained $\delta U_4 = J\partial_z J$, and so:

$$\begin{cases} U_4 = T^2 + J\partial_z J \\ \bar{U}_4 = \bar{T}^2 - \bar{J}\partial_{\bar{z}}\bar{J} \end{cases} \tag{18}$$

For $s = 5$, we get:

$$\Theta_4 = \partial_{\bar{z}}\left[\frac{3}{5}(\partial_z\phi)^5 + \partial_z\phi(\partial_z^2\phi)^2 - \frac{7}{2}(\partial_z\phi)^3\partial_z^2\phi - \frac{1}{2}\partial_z^2\phi\partial_z^3\phi - \frac{1}{2}\partial_z\phi\psi\partial_z^3\psi + \right.$$

$$\left. +\frac{1}{2}\partial_z^3\phi\psi\partial_z\psi + \frac{1}{2}\partial_z^2\phi\psi\partial_z^2\psi - \frac{13}{2}\partial_z\phi\partial_z^2\phi\psi\partial_z\psi + 2(\partial_z\phi)^3\psi\partial_z\psi - 3(\partial_z\phi)^2\psi\partial_z^2\psi\right].$$

Using similar arguments, it is straightforward to show that:

$$\begin{cases} U_6 = T^3 + 1/2(\partial_z T)^2 + 2TJ\partial_z J - 1/2J\partial_z^3 J \\ \bar{U}_6 = \bar{T}^3 + 1/2(\partial_{\bar{z}}\bar{T})^2 - 2\bar{T}\bar{J}\partial_{\bar{z}}\bar{J} + 1/2\bar{J}\partial_{\bar{z}}^3\bar{J} \end{cases} \tag{19}$$

We note that in the bosonic limit we recover exactly the IM of ref. [16].

2.2 The $B^{(1)}(0,1)$ theory

The $B^{(1)}(0,1)$ theory is defined by the superspace equation,

$$D_1 D_2 \Phi = ie^{2\Phi} - \frac{1}{2}\theta_1\theta_2 e^{-4\Phi}. \tag{20}$$

The second term on the right-hand side spoils invariance under supersymmetry. This is a common feature of Toda theories based on contragradient Lie superalgebras, [1] [5].

Alternatively, eq.(20) can be seen as the compatibilty condition for a linear system similar to (1), where this time the graded matrices take the form:

$$A_1(\lambda) = \begin{pmatrix} -2D_1\Phi & -i\lambda\sqrt{2} & 0 \\ 0 & 0 & i\lambda\sqrt{2} \\ -\lambda\theta_2 & 0 & 2D_1\Phi \end{pmatrix} \qquad A_2(\lambda) = \frac{1}{\lambda}\begin{pmatrix} 0 & 0 & \theta_1 e^{-4\Phi} \\ \sqrt{2}e^{2\Phi} & 0 & 0 \\ 0 & \sqrt{2}e^{2\Phi} & 0 \end{pmatrix}.$$

Expressing eq.(20) in components, we get in Euclidean space:

$$\begin{cases} F = -ie^{\phi}; \partial_z \bar{\psi} = -e^{\phi}\psi; \partial_{\bar{z}}\psi = -e^{\phi}\bar{\psi} \\ \partial_z \partial_{\bar{z}}\phi = e^{2\phi} - e^{\phi}\bar{\psi}\psi - \frac{1}{4}e^{-2\phi} \end{cases} \tag{21}$$

The bosonic limit of this theory is the $a_1^{(1)}$ bosonic Toda theory. It was conjectured [5] that the gaps in the sequence of conservation laws be periodic with period equal to 2. Specificaly, there will be an infinite set of conserved densities, $\partial_{\bar{z}}T_{s+1} = \partial_z \Theta_{s-1}$, with $s = 1, 3, 5, \cdots$

Considering the most general Ansatz, we obtained the following elements:

$$\begin{aligned} T_2 = & \ (\partial_z \phi)^2 - \partial_z \psi \psi \\ T_4 = & \ (\partial_z^2 \phi)^2 + (\partial_z \phi)^4 + 3(\partial_z \phi)^2 \psi \partial_z \psi + \partial_z \psi \partial_z^2 \psi + 3\partial_z \phi \partial_z^2 \psi \psi \\ \\ \Theta_0 = & \ e^{2\phi} - e^{\phi}\bar{\psi}\psi + \frac{1}{4}e^{-2\phi} \\ \Theta_2 = & \ 2(\partial_z \phi)^2 e^{2\phi} + 4(\partial_z \phi)^2 e^{\phi}\psi\bar{\psi} + \frac{1}{2}(\partial_z \phi)^2 e^{-2\phi} + 2e^{2\phi}\partial_z \psi \psi + \frac{3}{2}e^{-2\phi}\psi \partial_z \psi + \\ & \ + 2\partial_z \phi e^{\phi}\bar{\psi}\partial_z \psi \end{aligned} \tag{22}$$

3 The theories on the half-line

Let us assume a boundary located at $x = 0$. We will follow closely ref. [12].

The action on the half-line $x \in (-\infty, 0]$ is the sum of two contributions

$$S = S_0 + S_{\mathcal{B}} \equiv \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \left\{ \theta(-x)\mathcal{L}_0 + \delta(x)\mathcal{B}(\phi, \psi, \bar{\psi}) \right\},$$

where $\mathcal{L}_0$ is the bulk lagrangian density for either theory and the boundary potential $\mathcal{B}$ is assumed to be independent of the field derivatives. θ is the Heaviside step function.

Minimizing the action leads to the bulk field equations. Furthermore, we get the boundary conditions at $x = 0$:

$$\partial_x \phi + \frac{\partial \mathcal{B}}{\partial \phi} = 0, \qquad \psi - \frac{\partial \mathcal{B}}{\partial \psi} = 0, \qquad \bar{\psi} + \frac{\partial \mathcal{B}}{\partial \bar{\psi}} = 0. \tag{23}$$

3.1 The super-Liouville theory

We first investigate under what circumstances supersymmetry will be preserved in the presence of a boundary. It turns out, as we shall see, that it is still possible to keep half of the supersymmetries, provided one chooses suitable boundary conditions. These conditions equally preserve the conformal invariance of the theory and therefore its integrability.

Let us start by writing explicitly the variation of the action eq.(16) under the transformations (14) in the presence of the boundary.

$$\delta_S S_0 = \int_{-\infty}^{+\infty} dy \left\{ \frac{1}{2}(\eta\psi + \bar{\eta}\bar{\psi})\phi_x + (\bar{\psi}\eta + \psi\bar{\eta})e^{\phi} + \frac{i}{2}(\eta\psi - \bar{\eta}\bar{\psi})\phi_y \right\} |_{x=0}.$$

This expression can be compensated for by adding a boundary term. On dimensional grounds, we consider a boundary potential of the form:

$$\mathcal{B}_{\mathcal{S}} = c_S e^{\phi} + M_S \bar{\psi}\psi.$$

The boundary equations of motion arising from this term are:

$$\phi_x = -c_S e^{\phi}, \qquad \psi + M_S \bar{\psi} = 0, \qquad (M_S^2 = 1).$$

Under a supersymmetry transformation, we have:

$$\delta_S \int_{-\infty}^{+\infty} dy \mathcal{B}_S =$$
$$= \int_{-\infty}^{+\infty} dy \left\{ \tfrac{1}{2} M_S(\bar{\eta}\psi + \eta\bar{\psi})\phi_x + (c_S + M_S)(\eta\psi + \bar{\eta}\bar{\psi})e^{\phi} + \tfrac{i}{2} M_S(\bar{\eta}\psi + \bar{\psi}\eta)\phi_y \right\}.$$

It is only possible to keep half of the supersymmetries. We therefore choose $\bar{\eta} = \pm\eta$. The sum of the two contributions is thus:

$$\delta_S S_0 + \delta_S S_{\mathcal{B}_S} =$$
$$= \int_{-\infty}^{+\infty} dy \left\{ \tfrac{1}{2}(1 \pm M_S)\eta(\psi \pm \bar{\psi})\phi_x + (c_S + M_S \mp 1)\eta(\psi \pm \bar{\psi})e^{\phi} + \tfrac{i}{2}(1 \pm M_S)\eta(\psi \mp \bar{\psi})\phi_y \right\}.$$

The integrand in the above expression will be a total y-derivative, if we choose one of the following possibilities:

$$
\begin{aligned}
&1) \quad \phi_x = \mp 2e^{\phi}, \quad \psi \mp \bar{\psi} = 0 \\
&2) \quad \phi_x = \mp 2e^{\phi}, \quad \psi_y = \bar{\psi}_y = 0 \\
&3) \quad \phi_x = -c_S e^{\phi}, \quad \psi \pm \bar{\psi} = 0, \qquad \phi_y = 0
\end{aligned}
$$

It is easy to show that under an infinitesimal supersymmetry transformation,

$$
\begin{aligned}
&\delta_S(\phi_x + c_S e^{\phi}) = -i\eta\partial_y(\psi \mp \bar{\psi}) + (c_S \mp 2)\eta(\psi \pm \bar{\psi})e^{\phi}, \quad &&\delta_S(\phi_y) = \eta\partial_y(\psi \pm \bar{\psi}), \\
&\delta_S(\psi_y) = -\tfrac{\eta}{2}\partial_y(\phi_x \pm 2e^{\phi} - i\phi_y), &&\delta_S(\bar{\psi}_y) = \pm\tfrac{\eta}{2}\partial_y(\phi_x \pm 2e^{\phi} + i\phi_y), \\
&\delta_S(\psi \pm \bar{\psi}) = i\eta\phi_y, &&\delta_S(\psi \mp \bar{\psi}) = -\eta(\phi_x \pm 2e^{\phi}).
\end{aligned}
$$

If in 2) we take $\phi_y = 0$, the first two cases will be supersymmetry preserving, whilst the latter will not.

Let us consider additional terms in the boundary potential, $\epsilon_S \psi + \bar{\epsilon}_S \bar{\psi}$. Under a supersymmetry transformation such that $\bar{\eta} = \pm\eta$:

$$\delta_S(\epsilon_S \psi + \bar{\epsilon}_S \bar{\psi}) = -\frac{1}{2}(\epsilon_S \mp \bar{\epsilon}_S)\eta\phi_x \mp (\epsilon_S \mp \bar{\epsilon}_S)\eta e^{\phi} + \frac{i}{2}(\epsilon_S \pm \bar{\epsilon}_S)\eta\phi_y.$$

This will be a total y-derivative if

$$
\begin{aligned}
&1) \qquad \epsilon_S \mp \bar{\epsilon}_S = 0 \\
&2) \qquad \epsilon_S \pm \bar{\epsilon}_S = 0, \qquad \phi_x = \mp 2e^{\phi}
\end{aligned}
$$

In summary, the boundary potential,

$$\mathcal{B}_S = \pm 2e^{\phi} + M_S \bar{\psi}\psi + \epsilon_S \psi + \bar{\epsilon}_S \bar{\psi}, \tag{24}$$

restores supersymmetry, provided:

$$
\begin{aligned}
&1) \quad M_S = \mp 1, \quad \bar{\epsilon}_S = \pm\epsilon_S, \quad \psi \mp \bar{\psi} = -\epsilon_S \\
&2) \quad M_S^2 \neq 1, \quad \bar{\epsilon}_S = \mp\epsilon_S, \quad \phi_y = 0 \\
&\qquad\qquad\qquad \psi = -\tfrac{\epsilon_s}{1 \mp M_S}, \quad \bar{\psi} = \tfrac{\pm\epsilon}{1 \mp M_S}
\end{aligned}
$$

Let us now discuss the integrability of the theory. According to Cardy, [14], invariance of the boundary conditions under a symmetry generated by some set of conserved currents

$(W^{(r)}, \bar{W}^{(r)})$ requires $W^{(r)} = \bar{W}^{(r)}$ on the boundary. If we take $W^{(r)}$ to be the stress tensor, we get from eqs.(12), (13):

$$-i\phi_x\phi_y - \frac{1}{2}\psi_x\psi + \frac{i}{2}\psi_y\psi + i\phi_{xy} = \frac{1}{2}\bar{\psi}_x\bar{\psi} + \frac{i}{2}\bar{\psi}_y\bar{\psi}.$$

The bosonic part, $-i\phi_x\phi_y + i\phi_{xy}$, vanishes for $\phi_x = ce^\phi$, in which case :

$$\psi_y\psi = \bar{\psi}_y\bar{\psi}, \tag{25}$$

where we used the equations of motion (10) to eliminate the x-derivatives. There are two solutions to eq.(25):

$$\begin{array}{ll} 1) & \bar{\psi} = \pm\psi \\ 2) & \psi_y = \bar{\psi}_y = 0 \end{array} \tag{26}$$

We notice that these solutions are reminescent of the conditions obtained for the conservation of supersymmetry. However, c and ϕ_y remain arbitrary. But we can still impose similar constraints on the supercurrents $(J, \bar{J})$ and this should fix c and ϕ_y. The boundary condition is $\bar{\eta}\bar{\psi} = \eta\psi$. Remember that we want to keep half of the supersymmetries, by setting $\bar{\eta} = \pm\eta$. Accordingly, we impose the boundary condition $\bar{J} = \mp J$ and get:

$$(\phi_x + i\phi_y \pm 2e^\phi)\bar{\psi} - 2i\bar{\psi}_y = \mp(\phi_x - i\phi_y \pm 2e^\phi)\psi \mp 2i\psi_y, \tag{27}$$

where we used the explicit expressions (17) for J and $\bar{J}$. Let us use as Ansatz the conditions (26) obtained above for the conservation of conformal invariance. We then get:

$$\begin{array}{lll} 1) & \bar{\psi} = \pm\psi, & \phi_x = \mp 2e^\phi \\ 2) & \psi_y = \bar{\psi}_y = 0, & \phi_x = \mp 2e^\phi, & \phi_y = 0 \end{array} \tag{28}$$

In 1) the sign of $\bar{\psi}$ was chosen so as to cancel the terms proportional to ψ_y and $\bar{\psi}_y$ in eq.(27). It is easy to check that these conditions also preserve the following combinations of the IM:

$$I_s = \int_{-\infty}^0 dx(U_{s+1} + \bar{U}_{s+1}), \qquad (s = 1, 3, 5, \cdots).$$

We just have to show that $U_{s+1} = \bar{U}_{s+1}$ at $x = 0$ as a consequence of the stress tensor and the supercurrent satisfying $T = \bar{T}$ and $\bar{J} = \mp J$.

All polynomials T^n $(n > 1)$ automatically satisfy $T^n = \bar{T}^n$. The first non-trivial term is $(\partial_z T)^2$. From the conservation of the stress tensor, we have $T_x = -iT_y$, $\bar{T}_x = i\bar{T}_y$. This implies that at $x = 0$,

$$(\partial_z T)^2 = -T_y^2 = -\bar{T}_y^2 = (\partial_{\bar{z}}\bar{T})^2.$$

Similarly, from $J_x = -iJ_y$, $\bar{J}_x = i\bar{J}_y$, we have:

$$J\partial_z J = -iJJ_y = -i(\mp\bar{J})(\mp\bar{J}_y) = -i\bar{J}\bar{J}_y = -\bar{J}\partial_{\bar{z}}\bar{J}.$$

Altogether, this means that $U_4 = \bar{U}_4$. Next we consider the term $J\partial_z^3 J$. We use the following identities,

$$\begin{cases} J_{xyy} = & iJ_{xxy} = & -J_{xxx} = & -iJ_{yyy} \\ \bar{J}_{xyy} = & -i\bar{J}_{xxy} = & -\bar{J}_{xxx} = & i\bar{J}_{yyy} \end{cases}$$

to show that

$$J\partial_z^3 J = \frac{1}{8}J(J_{xxx} - 3iJ_{xxy} - 3J_{xyy} + iJ_{yyy}) = iJJ_{yyy} = i\bar{J}\bar{J}_{yyy} = -\bar{J}\partial_{\bar{z}}^3\bar{J}.$$

Again, we have $U_6 = \bar{U}_6$. As advertised, the conditions (28) coincide precisely with the conditions for conservation of supersymmetry, provided we take $\epsilon_S = 0$. In summary, we found that the conditions preserving half of the supersymmetries in the surface configuration, also ensure the conservation of the superconformal invariance and the IM. This is not surprising, since the IM, being composite fields of the stress tensor, the supercurrents and their derivatives, are deeply connected with the superconformal symmetry of the theory.

3.2 The $B^{(1)}(0,1)$ theory

Suppose that the boundary potential $\mathcal{B}$ can be chosen in such a way that at $x = 0$

$$T_{s+1} - \bar{T}_{s+1} - (\Theta_{s-1} - \bar{\Theta}_{s-1}) = \frac{d}{dy}\Sigma_s(y), \tag{29}$$

where $\Sigma_s(y)$ is some functional of the boundary fields. Then the 'spin' s charge given by

$$Q_s = \int_{-\infty}^{0} dx(T_{s+1} + \bar{T}_{s+1} + \Theta_{s-1} + \bar{\Theta}_{s-1}) - i\Sigma_s(y)$$

is a non-trivial IM, [11]. Let us now look for potentials that produce expressions like eq.(29). From the equations of motion, we have:

$$\begin{cases} \psi_{xy} = -2e^{\phi}\bar{\psi}_y - 2\phi_y e^{\phi}\bar{\psi} - i\psi_{yy} \\ \psi_{xx} = 4e^{2\phi}\psi - 2\phi_x e^{\phi}\bar{\psi} + 2i\phi_y e^{\phi}\bar{\psi} - \psi_{yy} \\ \bar{\psi}_{xy} = -2e^{\phi}\psi_y - 2\phi_y e^{\phi}\psi + i\bar{\psi}_{yy} \\ \bar{\psi}_{xx} = 4e^{2\phi}\bar{\psi} - 2\phi_x e^{\phi}\psi - 2i\phi_y e^{\phi}\psi - \bar{\psi}_{yy} \end{cases}$$

$$\phi_{xx} = 4e^{2\phi} - \phi_{yy} - 4e^{\phi}\bar{\psi}\psi - e^{-2\phi}.$$

Moreover, from the eq.(23), we have at $x = 0$:

$$\frac{\partial^2 \mathcal{B}}{\partial\phi\partial\psi} = \frac{\partial^2 \mathcal{B}}{\partial\phi\partial\bar{\psi}} = \frac{\partial^2 \mathcal{B}}{\partial\psi\partial\bar{\psi}} = 0.$$

Consequently,

$$\phi_{xy} = -\frac{\partial^2 \mathcal{B}}{\partial\phi^2}\phi_y.$$

Using these expressions, we get:

$$T_4 - \bar{T}_4 + \bar{\Theta}_2 - \Theta_2 = \mathcal{W}_b + \mathcal{W}_R,$$

where $\mathcal{W}_b$ is a purely bosonic contribution,

$$\mathcal{W}_b = -\frac{i}{4}\frac{\partial^2 \mathcal{B}}{\partial\phi^2}\phi_{yy}\phi_y - \frac{i}{8}\frac{\partial \mathcal{B}}{\partial\phi}\phi_y^3 + \frac{i}{2}\left\{\frac{1}{4}\left(\frac{\partial \mathcal{B}}{\partial\phi}\right)^3 + \frac{\partial^2 \mathcal{B}}{\partial\phi^2}\left(e^{2\phi} - \frac{1}{4}e^{-2\phi}\right) - \frac{\partial \mathcal{B}}{\partial\phi}\left(e^{2\phi} + \frac{1}{4}e^{-2\phi}\right)\right\}\phi_y.$$

We look for solutions of the form $\mathcal{B}(\phi, \psi, \bar{\psi}) = \mathcal{B}_b(\phi) + \mathcal{B}_f(\psi, \bar{\psi})$. It is straightforward to show that for $\mathcal{B}_b(\phi) = ae^{\phi} + be^{-\phi}$, where a and b are arbitrary constants, $\mathcal{W}_b$ will automatically be a total y-derivative. The remaining contribution $\mathcal{W}_R$ is given by:

$$\mathcal{W}_R = -\frac{i}{2}\bar{\psi}\psi\phi_y\left(\frac{\partial^2 \mathcal{B}_b}{\partial\phi^2} + \frac{\partial \mathcal{B}_b}{\partial\psi}\right)e^{\phi} - \frac{3i}{8}\left[\left(\frac{\partial \mathcal{B}_b}{\partial\phi}\right)^2 - \phi_y^2 - 4e^{2\phi} - e^{-2\phi}\right](\bar{\psi}_y\bar{\psi} - \psi_y\psi) +$$

$$+\frac{3}{4}\frac{\partial \mathcal{B}_b}{\partial\phi}\phi_y(\bar{\psi}\psi_y + \psi\bar{\psi}_y) + \frac{i}{2}\frac{\partial \mathcal{B}_b}{\partial\phi}e^{\phi}(\psi_y\bar{\psi} - \bar{\psi}_y\psi) + 2\phi_y e^{\phi}(\psi_y\bar{\psi} + \bar{\psi}_y\psi) +$$

$$+i(\bar{\psi}_{yy}\bar{\psi}_y - \psi_{yy}\psi_y) + e^{\phi}(\bar{\psi}\psi_{yy} - \bar{\psi}_{yy}\psi) + \frac{3}{4}\frac{\partial \mathcal{B}_b}{\partial\phi}(\psi_{yy}\psi + \bar{\psi}_{yy}\bar{\psi}) + \frac{3i}{4}\phi_y(\psi_{yy}\psi - \bar{\psi}_{yy}\bar{\psi}).$$

Because $\psi, \bar\psi$ are Grassmann variables, $\mathcal{B}_f$ takes the form $\mathcal{B}_f(\psi,\bar\psi) = M\bar\psi\psi + \epsilon\psi + \bar\epsilon\bar\psi$, where M, ϵ, $\bar\epsilon$ are constant parameters, M being bosonic and the remaining fermionic. From eq.(23), we have the following possibilities at $x = 0$:

$$
\begin{array}{llll}
1) & \psi = -\frac{\epsilon+M\bar\epsilon}{1-M^2}, & \bar\psi = \frac{\bar\epsilon+M\epsilon}{1-M^2}, & M \neq \pm 1 \\
2) & \bar\psi = \mp(\psi+\epsilon), & \bar\epsilon = \mp\epsilon, & M = \pm 1
\end{array}
$$

In the first case, $\mathcal{W}_R$ is automatically a total y-derivative, irrespective of the values of $a, b, \epsilon, \bar\epsilon$ and $M(\neq \pm 1)$. In the latter case, we get $\epsilon = \bar\epsilon = 0$ and $a = \mp 2$, corresponding to $M = \pm 1$. In summary, there will be a spin $s = 3$ conserved charge in the following cases:

$$
\begin{array}{l}
1) \quad \mathcal{B}(\phi,\psi,\bar\psi) = ae^{\phi} + be^{-\phi} + M\bar\psi\psi + \epsilon\psi + \bar\epsilon\bar\psi \\
\quad\quad \phi_x = -ae^{\phi} + be^{-\phi}, \quad \psi = -\frac{\epsilon+M\bar\epsilon}{1-M^2}, \quad \bar\psi = \frac{\bar\epsilon+M\epsilon}{1-M^2}, \\
\quad\quad\quad a, b, \epsilon, \bar\epsilon \text{ and } M(\neq \pm 1) \text{ are arbitrary.} \\
2) \quad \mathcal{B}(\phi,\psi,\bar\psi) = \mp 2e^{\phi} + be^{-\phi} \pm \bar\psi\psi \\
\quad\quad \phi_x = \pm 2e^{\phi} + be^{-\phi}, \quad \psi \pm \bar\psi = 0 \\
\quad\quad\quad b \text{ is arbitrary.}
\end{array}
$$

4 Conclusions

Let us restate our results. We derived a recursive formula for an infinity of integrals of motion (IM) for the super-Liouville-equation (SLE) which consist of a supersymmetric extension of the classical expressions in ref.[16]. These IM and its eigenstates, together with the factorisable S-matrix constitute an alternative description of the conformal theory.

The boundary equations of motion, preserving half of the supersymmetries, automatically conserve the superconformal invariance and half of the IM on the half-line. Furthermore, these boundary conditions are unambiguously determined, i.e. there are no unfixed parameters. A similar situation occurs in the super-sine-Gordon theory [12] and appears to be a consequence of supersymmetry. Indeed, our analysis of the non-supersymmetric $B^{(1)}(0,1)$ theory reveals that, in contrast with the two models above, the boundary potential depends on free parameters.

Acknowledgements

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